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THREE-DIMENSIONAL VIBRATION OF RECTANGULAR PLATES : VARIANCE OF SIMPLE SUPPORT CONDITIONS AND INFLUENCE OF IN-PLANE INERTIA

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Abstract—Plate models developed on the basis of the three-dimensional elasticity theory are crucial to the correct solutions in dynamics problems. In a three-dimensional setting, there exist numerous possible definitions of boundary conditions. This paper attempts to identify the variance of three-dimensional simple support conditions existing in practice. The *hard* simple support condition assumes vanishing normal stresses at the edges and zero transverse and tangential displacement at the peripheries. The *soft* simple support, on the other hand, imposes zero transverse displacement with vanishing normal and shear stresses at the boundaries. To quantify the relative effects of each boundary condition on the vibratory responses of thick plates, a global three-dimensional Ritz formulation is employed for analysis. This technique uses sets of finite polynomial series in Cartesian co-ordinates to approximate the global normal mode variations of the plate. These separable one-dimensional polynomials are orthogonally generated within the plate domain such that all the essential edge conditions are identically satisfied. The accuracy of the proposed three-dimensional Ritz method is assured from the convergence and comparison studies. Numerical experiments have been conducted to study firstly, the effects of geometric parameters on the overall normal mode characteristics of simply supported plates ; and secondly, the effects of in-plane inertia on the vibration frequencies of plates with different thicknesses. The three-dimensional deformed mode shapes for selected cases have also been computed. These have served to describe more vividly the normal mode characteristics of different types of simply supported rectangular plates.

1. INTRODUCTION

To achieve accurate and correct solutions to the static and dynamics problems in mechanics requires the use of three-dimensional elasticity theory. Prominent researchers such as Srinivas and Rao (1970), Srinivas (1972), Wittrick (1987) and Hutchinson and Zillmer (1983) have demonstrated the importance of three-dimensional elasticity solutions in providing benchmark references to the various refined theories and the existing finite element codes. Savoia and Reddy (1992), using a layer-wise three-dimensional elasticity formulation, have uncovered several interesting properties of cross-ply and angle-ply laminated plates under transverse loading. Hutchinson (1981), on the other hand, by comparing his three-dimensional frequency solutions with the Timoshenko beam approximations at different shear correction factors, was able to probe in-depth the influence of the shear coefficient on the dynamics properties of beams of different sizes. These and several emerging researches (Liew *et al.*, 1993, 1994, 1995a,b, 1996) in this direction, have pointed convincingly to the comparative advantages of three-dimensional elasticity formulation for problems in engineering mechanics.

In a three-dimensional setting, there exist numerous possible definitions of boundary conditions. In this paper, we attempt to identify the variance of the three-dimensional simple support conditions existing in practice. The *hard* simple support condition presumes vanishing normal stresses at the edges and zero transverse and tangential displacements at the peripheries. The *soft* simple support condition, on the other hand, imposes zero transverse displacement with vanishing normal and shear stresses at the edges. These two sets of simple support conditions address some of the most commonly encountered support conditions in engineering practice.

In this study, the influence of geometric parameters on the normal mode characteristics of simply supported plates is investigated. Particular attention is drawn to the influence of in-plane inertia on the vibration frequencies of both thin and thick plates. These solutions may serve to establish the panel flutter behaviours of aerodynamic bodies, the resonant responses of critical machine components and the periodic load bearing capability of civil engineering structures.

In order to obtain solutions for the above studies, an accurate and efficient three-dimensional Ritz algorithm is developed for the analysis. The method uses a linear, small-strain, three-dimensional elasticity theory and the Ritz minimum energy principle to derive the governing eigenvalue equation. The surface and thickness variations in three dimensions are approximated in this case by sets of one-dimensional finite polynomial series. These functions are orthogonally generated and have been proven to be highly efficient and accurate in many numerical applications (Lam and Hung, 1990). The technique yields natural frequencies and deflection mode shapes for thick rectangular plates. These results aim to establish better normal mode characterizations of plates with hard and soft simple support conditions. The influence of in-plane inertia on the vibration response of plates at different thickness ratios can also be deduced from this study.

2. THEORETICAL FORMULATION

2.1. *Preliminary definitions*

Figure 1 shows a thick, homogeneous, isotropic rectangular plate of uniform thickness, h. The plate dimensions are defined on a right-handed orthogonal co-ordinate system (x_1, x_2, x_3) with the origin at the geometric centre of the plate. The reference plane containing x_1 and x_2 lies symmetrically between the top and bottom surfaces of the plate. The plate domain is bounded by $-a/2 \leqslant x_1 \leqslant a/2$, $-b/2 \leqslant x_2 \leqslant b/2$ and $-h/2 \leqslant x_3 \leqslant h/2$. At a general point, the spatial displacement is resolved into the in-plane (u_1, u_2) and out-of-plane (u_3) components, respectively. The plates treated in this paper are assumed to be stress free at the top and bottom surfaces.

2.2. *Energy functional in three-dimensional space*

The strain energy in a three-dimensional setting has the following form (Hung *et al.*, 1994) :

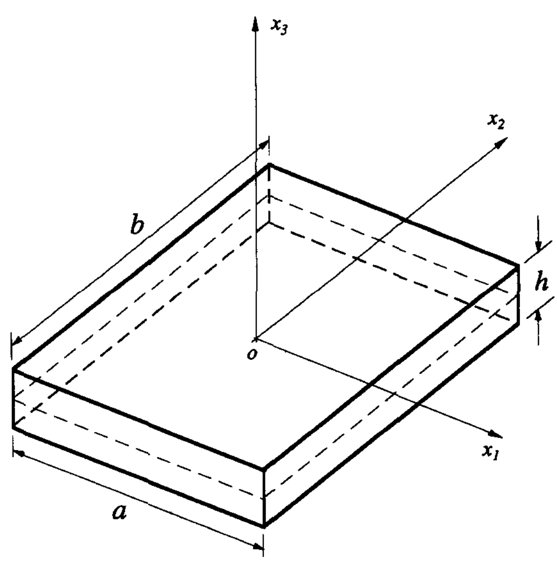


Fig. 1. Dimension and geometry of a thick rectangular plate.

$$V_{\max} = \frac{1}{2}\iiint_v \left[\lambda'\left(\sum_{i=1}^{3} U_{i,i}\right)^2 + G\{(U_{1,2}+U_{2,1})^2+(U_{2,3}+U_{3,2})^2+ (U_{1,3}+U_{3,1})^2\} + 2G\left(\sum_{i=1}^{3} U_{i,i}^2\right)\right] \mathrm{d}v \; ; \quad i = 1,2,3 \tag{1}$$

where $(\,)_{,i} = \partial(\,)/\partial x_i$ and the Lamé constants, λ' and G (shear modulus), are defined as

$$\lambda' = \frac{\nu E}{[(1+\nu)(1-2\nu)]} \tag{2a}$$

$$G = \frac{E}{2(1+\nu)} \tag{2b}$$

where E is the modulus of elasticity and ν is the Poisson ratio.

The maximum kinetic energy, on the other hand, is given by

$$T_{\max} = \frac{\rho\omega^2}{2}\iiint_v U_i^2 \, \mathrm{d}v \; ; \quad i = 1,2,3 \tag{3}$$

in which ρ is the mass density per unit volume and ω is the angular frequency.

The energy functional consisting of the maximum strain and kinetic energies is defined as

$$\Pi = V_{\max} - T_{\max}. \tag{4}$$

3. METHOD OF SOLUTION

3.1. *Governing eigenvalue equation*

The displacement amplitude functions in eqns (1) and (3) are assumed in the following forms

$$U_i = \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{l=1}^{L} C_{mnl}^{i}\, {}^{i}\phi_m(x_1)\,{}^{i}\phi_n(x_2)\,{}^{i}\phi_l(x_3), \quad i = 1,2,3. \tag{5}$$

In the above expression, C_{mnl}^{i}, are the undetermined coefficients and ${}^{i}\phi(x_1)$, ${}^{i}\phi(x_2)$, ${}^{i}\phi(x_3)$, are the corresponding one-dimensional polynomial functions for the surface and thickness variations of the plate.

Expanding eqn (4) according to eqns (1)–(3) and (5) gives the energy functional which is minimized according to the Ritz principle

$$\frac{\partial \Pi}{\partial C_{mnl}^{i}} = 0 \; ; \quad i = 1,2,3 \tag{6}$$

to yield the following linear eigenvalue equation in three dimensions:

$$(\mathbf{K} - \Omega^2\mathbf{M})\{\mathbf{C}\} = \{\mathbf{0}\} \tag{7}$$

where $\{\mathbf{C}\} = \{C^1\; C^2\; C^3\}^{\mathrm{T}}$ are the eigenvectors and $\Omega = \omega a(\rho/E)^{1/2}$, the corresponding eigenvalues.

The global stiffness $\mathbf{K}$ and mass $\mathbf{M}$ matrices have the following forms

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}^{11} & \mathbf{k}^{12} & \mathbf{k}^{13} \\ & \mathbf{k}^{22} & \mathbf{k}^{23} \\ Sym & & \mathbf{k}^{33} \end{bmatrix} \tag{8a}$$

and

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}^{11} & 0 & 0 \\ & \mathbf{m}^{22} & 0 \\ Sym & & \mathbf{m}^{33} \end{bmatrix} \tag{8b}$$

in which the corresponding constitutive matrices are given by

$$\mathbf{k}^{11}_{minjlk} = \frac{1-v}{(1-2v)}(E^{11}_{mi}F^{00}_{nj}G^{00}_{lk})_{11} + \frac{1}{2}\left(\frac{a}{b}\right)^2 (E^{00}_{mi}F^{11}_{nj}G^{00}_{lk})_{11} + \frac{1}{2}\left(\frac{a}{h}\right)^2 (E^{00}_{mi}F^{00}_{nj}G^{11}_{lk})_{11} \tag{9a}$$

$$\mathbf{k}^{12}_{minjlk} = \left(\frac{a}{b}\right)\left\{\frac{v}{(1-2v)}(E^{10}_{mi}F^{01}_{nj}G^{00}_{lk})_{12} + \frac{1}{2}(E^{01}_{mi}F^{10}_{nj}G^{00}_{lk})_{12}\right\} \tag{9b}$$

$$\mathbf{k}^{13}_{minjlk} = \left(\frac{a}{h}\right)\left\{\frac{v}{(1-2v)}(E^{10}_{mi}F^{00}_{nj}G^{01}_{lk})_{13} + \frac{1}{2}(E^{01}_{mi}F^{00}_{nj}G^{10}_{lk})_{13}\right\} \tag{9c}$$

$$\mathbf{k}^{22}_{minjlk} = \left(\frac{a}{b}\right)^2\left\{\frac{1-v}{(1-2v)}(E^{00}_{mi}F^{11}_{nj}G^{00}_{lk})_{22}\right\} + \frac{1}{2}\left(\frac{a}{h}\right)^2 (E^{00}_{mi}F^{00}_{nj}G^{11}_{lk})_{22} + \frac{1}{2}(E^{11}_{mi}F^{00}_{nj}G^{00}_{lk})_{22} \tag{9d}$$

$$\mathbf{k}^{23}_{minjlk} = \left(\frac{a^2}{bh}\right)\left\{\frac{v}{(1-2v)}(E^{00}_{mi}F^{10}_{nj}G^{01}_{lk})_{23} + \frac{1}{2}(E^{00}_{mi}F^{01}_{nj}G^{10}_{lk})_{23}\right\} \tag{9e}$$

$$\mathbf{k}^{33}_{minjlk} = \left(\frac{a}{h}\right)^2\left\{\frac{1-v}{(1-2v)}(E^{00}_{mi}F^{00}_{nj}G^{11}_{lk})_{33}\right\} + \frac{1}{2}\left(\frac{a}{b}\right)^2 (E^{00}_{mi}F^{11}_{nj}G^{00}_{lk})_{33} + \frac{1}{2}(E^{11}_{mi}F^{00}_{nj}G^{00}_{lk})_{33} \tag{9f}$$

$$\mathbf{m}^{11}_{minjlk} = (1+v)(E^{00}_{mi}F^{00}_{nj}G^{00}_{lk})_{11} \tag{9g}$$

$$\mathbf{m}^{22}_{minjlk} = (1+v)(E^{00}_{mi}F^{00}_{nj}G^{00}_{lk})_{22} \tag{9h}$$

$$\mathbf{m}^{33}_{minjlk} = (1+v)(E^{00}_{mi}F^{00}_{nj}G^{00}_{lk})_{33}. \tag{9i}$$

The products of integrals in eqns (9a–i) are given by

$$(E^{rs}_{mi})_{\alpha\beta} = \int_{-0.5}^{0.5}\left[\frac{\partial^r\{{}^{\alpha}\phi_m(\bar{x}_1)\}}{\partial \bar{x}_1^r}\right]\left[\frac{\partial^s\{{}^{\beta}\phi_i(\bar{x}_1)\}}{\partial \bar{x}_1^s}\right]\mathrm{d}\bar{x}_1 \tag{10a}$$

$$(F^{rs}_{nj})_{\alpha\beta} = \int_{-0.5}^{0.5}\left[\frac{\partial^r\{{}^{\alpha}\phi_n(\bar{x}_2)\}}{\partial \bar{x}_2^r}\right]\left[\frac{\partial^s\{{}^{\beta}\phi_j(\bar{x}_2)\}}{\partial \bar{x}_2^s}\right]\mathrm{d}\bar{x}_2 \tag{10b}$$

$$(G^{rs}_{lk})_{\alpha\beta} = \int_{-0.5}^{0.5}\left[\frac{\partial^r\{{}^{\alpha}\phi_l(\bar{x}_3)\}}{\partial \bar{x}_3^r}\right]\left[\frac{\partial^s\{{}^{\beta}\phi_k(\bar{x}_3)\}}{\partial \bar{x}_3^s}\right]\mathrm{d}\bar{x}_3 \tag{10c}$$

where the variables, $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ in the above integrations are non-dimensionalized ;

$$\bar{x}_1 = \frac{x_1}{a}; \quad \bar{x}_2 = \frac{x_2}{b} \quad \text{and} \quad \bar{x}_3 = \frac{x_3}{h}. \tag{11}$$

3.2. *Admissible displacement functions*

The one-dimensional polynomial based displacement functions are intrinsically the products of (i) a basic function chosen to satisfy the essential geometric boundary conditions of the plate ; and (ii) a mathematically complete one-dimensional polynomial space. These functions are constructed from the Gram–Schmidt recurrence formula (Chihara, 1978).

For $P_k(\bar{x}) \in \{{}^i\phi_k(\bar{x})\ ;\ i = 1, 2, 3\}$ and $\bar{x} \in \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, the recurrence process gives

$$P_{k+1}(\bar{x}) = \{g(\bar{x}) - \Xi_k^A\}P_k(\bar{x}) - \Xi_k^B P_{k-1}(\bar{x})\ ; \quad k = 1, 2, 3 \ldots \tag{12}$$

The polynomial $P_0(\bar{x})$ is defined as zero and the constants Ξ_k^A and Ξ_k^B are chosen such that the polynomials generated in eqn (12) satisfy the orthogonality condition :

$$\int_{-0.5}^{0.5} P_m(\bar{x})P_n(\bar{x})\,\mathrm{d}\bar{x} = \begin{cases} \Lambda_{mn} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}. \tag{13}$$

The value of Λ_{mn} depends on the normalization used.

From the recurrence relation of eqn (12) and considering eqn (13), we have

$$\Xi_k^A = \Delta_1^k / \Delta_2^k \tag{14a}$$

$$\Xi_k^B = \Delta_2^k / \Delta_2^{k-1} \tag{14b}$$

with

$$\Delta_1^k = \int_{-0.5}^{0.5} g(\bar{x})P_k^2(\bar{x})\,\mathrm{d}\bar{x} \tag{15a}$$

$$\Delta_2^k = \int_{-0.5}^{0.5} P_k^2(\bar{x})\,\mathrm{d}\bar{x} \tag{15b}$$

$$\Delta_2^{k-1} = \int_{-0.5}^{0.5} P_{k-1}^2(\bar{x})\,\mathrm{d}\bar{x} \tag{15c}$$

and $g(\bar{x})$, the generating function in eqn (12), is chosen for the higher terms to satisfy the essential geometric boundary conditions.

The boundary conditions of the plate are uniquely satisfied by the basic functions, ${}^i\phi_1(\bar{x})$. In addition, these functions also take into account the symmetry inherent in the mode shapes for rectangular planform so as to reduce the determinant size of the eigenvalue equation. In this study, two types of simple support conditions and four distinct classes of symmetry modes are considered. These are discussed in the following section.

(a) *Hard simple support*. The hard simple support condition assumes vanishing normal stresses at the edges and zero transverse and tangential displacements at the peripheries :

$$\sigma_n = 0, \quad u_t = 0 \quad \text{and} \quad u_3 = 0. \tag{16a}$$

This is translated into the following kinematic constraints to be satisfied by the displacement amplitude functions at the edges

Table 1(a). The basic function in the $\bar{x}_1$ direction for the two types of simple supports

Boundary condition	$f_0(\bar{x}_1)$	$f_e(\bar{x}_1)$
Hard simple support at both ends ($\sigma_n = 0$, $u_t = 0$ and $u_3 = 0$)	${}^1\phi_1(\bar{x}_1) = \bar{x}_1$ ${}^2\phi_1(\bar{x}_1) = \bar{x}_1^3 - 0.25\bar{x}_1$ ${}^3\phi_1(\bar{x}_1) = \bar{x}_1^3 - 0.25\bar{x}_1$	${}^1\phi_1(\bar{x}_1) = 1.0$ ${}^2\phi_1(\bar{x}_1) = \bar{x}_1^2 - 0.25$ ${}^3\phi_1(\bar{x}_1) = \bar{x}_1^2 - 0.25$
Soft simple support at both ends ($\sigma_n = 0$, $\sigma_t = 0$ and $u_3 = 0$)	${}^1\phi_1(\bar{x}_1) = \bar{x}_1$ ${}^2\phi_1(\bar{x}_1) = \bar{x}_1$ ${}^3\phi_1(\bar{x}_1) = \bar{x}_1^3 - 0.25\bar{x}_1$	${}^1\phi_1(\bar{x}_1) = 1.0$ ${}^2\phi_1(\bar{x}_1) = 1.0$ ${}^3\phi_1(\bar{x}_1) = \bar{x}_1^3 - 0.25$

Table 1(b). Basic functions for U_1, U_2, U_3 components at each symmetry class

Symmetry class	$U_1(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ ${}^1\phi_1(\bar{x}_1)$	$U_1(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ ${}^1\phi_1(\bar{x}_2)$	$U_2(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ ${}^2\phi_1(\bar{x}_1)$	$U_2(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ ${}^2\phi_1(\bar{x}_2)$	$U_3(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ ${}^3\phi_1(\bar{x}_1)$	$U_3(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ ${}^3\phi_1(\bar{x}_2)$
SS	f_o	f_e	f_e	f_o	f_e	f_e
SA	f_o	f_o	f_e	f_e	f_e	f_o
AS	f_e	f_e	f_o	f_o	f_o	f_e
AA	f_e	f_o	f_o	f_e	f_o	f_o
Generating function $g(\bar{x}_1\,;\bar{x}_2) = (\bar{x}_1^2\,;\bar{x}_2^2)$						

$$U_t = 0\,; \quad U_n \neq 0 \quad \text{and} \quad U_3 = 0. \tag{16b}$$

(b) *Soft simple support.* The soft simple support condition imposes zero transverse displacement with vanishing normal and shear stresses at the boundaries :

$$\sigma_n = 0, \quad \sigma_t = 0 \quad \text{and} \quad u_3 = 0. \tag{17a}$$

In terms of displacement amplitude functions this is expressed as

$$U_t \neq 0\,; \quad U_n \neq 0 \quad \text{and} \quad U_3 = 0. \tag{17b}$$

(c) *Symmetry classes.* The basic functions are grouped into four symmetry classes about the x_1x_3-plane and the x_2x_3-plane in order to maximize the computational efficiency. Tables 1(a) and 1(b) summarize the symmetry classes and the corresponding basic and generating functions used for each type of simple support conditions. In Table 1(b), the symbols SS, SA, AS and AA denote symmetry–symmetry, symmetry–antisymmetry, antisymmetry–symmetry and antisymmetry–antisymmetry modes about the x_1x_3- and x_2x_3-planes, respectively. In the Ritz formulation, it is sufficient to choose basic functions that satisfy only the essential geometric boundary conditions of the plate.

Additional saving in terms of computational time and storage space can be achieved by further dividing the aforementioned symmetry modes into the symmetric thickness modes and the antisymmetric thickness modes. For symmetric thickness modes, the basic functions in the thickness direction for each displacement component are

$${}^j\phi_1(\bar{x}_3) = 1.0, \quad j = 1,2 \quad \text{and} \quad {}^3\phi_1(\bar{x}_3) = \bar{x}_3. \tag{18}$$

For the antisymmetric thickness modes, the corresponding basic functions are

$${}^j\phi_1(\bar{x}_3) = \bar{x}_3, \quad j = 1,2 \quad \text{and} \quad {}^3\phi_1(\bar{x}_3) = 1.0. \tag{19}$$

For both cases, the generating function $g(\bar{x}_3)$ is taken to be $\bar{x}_3^2$.

4. RESULTS AND DISCUSSION

Frequency results for plates with two types of simple supports have been computed. These serve to establish the dynamics behaviour of plates with different boundary conditions and dimensions. The frequency parameter is expressed in the non-dimensionalized form as

$$\lambda = (\omega b^2/\pi^2)(\rho h/D)^{1/2} \tag{20}$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity often used for plate analysis.

4.1. *Convergence characteristics*

To establish the degrees of accuracy of the solutions, convergence tests have been carried out for plates with hard and soft simple support conditions. Tables 2(a) and 2(b) show the variation of frequency parameters, λ, at each polynomial degree set. It is observed that the frequency parameters, λ, converge to four significant figures at a polynomial degree set of $6\times 6\times 4$ (determinant size of 432) for the thin ($h/b = 0.01$) and thick ($h/b = 0.25$) plates, respectively. Both the hard and soft simply supported plates require approximately the same number of terms in the deflection series for converged solutions. This speedy convergence is possible through the use of various symmetry considerations in the present method.

4.2. *Influence of geometric parameters and boundary conditions*

Table 3 quantifies the relative effects of plate thickness ratio (h/b) and different simple support conditions on the normal mode frequencies of rectangular plates. The plate thickness ratios are assumed at 0.01, 0.1, 0.25 and 0.5, respectively. It is observed that the increase in the thickness ratio leads to lowering of the frequency parameters for both hard and soft simply supported plates. A closer scrutiny of Table 3 shows that at small thickness ratio ($h/b = 0.01$), the frequencies are in good agreement for both types of simple supports. The deviation, however, widens as the thickness ratio increases and the through thickness contributions to stiffness and inertia become significant. This observation agrees well with those reported by Babuška and Li (1992). They have provided various theorems to prove that as the thickness becomes negligible, the discrepancy between the hard and soft supports disappears.

There are other mechanical attributes which follow the thickening of plates. It is observed that, as the plate thickness increases, the symmetric thickness modes (which exhibit predominant surface parallel motions) become dominant. For thin plate ($h/b = 0.01$), on the other hand, the antisymmetric thickness modes (which are predominantly out-of-plane motions) dominate in the lower vibration spectrum. To characterize more vividly the normal mode behaviours of these simply supported plates, three-dimensional mode shapes at each thickness ratio are presented in Figs 2 and 3 for hard and soft simply supported plates, respectively. The presence of surface parallel, symmetric thickness modes in the lower vibration spectrum for thick plates is evident in these diagrams. These surface parallel modes at the lower vibration spectrum restrict the applicability of the refined plate theory such as the Mindlin theory to the analysis of moderately thick plate. Since the first order Mindlin theory only considers flexural modes and in this study, it is found that the symmetric thickness modes (which include thickness shear, thickness twist and pure in-plane motions) often precede some of the flexural modes in thick plate vibrations.

4.3. *Influence of in-plane inertia on frequencies*

A numerical experiment is carried out to examine the influence of in-plane inertia on the normal modes of simply supported plates. This is done by eliminating the in-plane (u_1, u_2) contributions to the kinetic energy expression.

Table 2(a). Convergence of frequency parameters, $\lambda = \omega b^2/\pi^2(\rho h/D)^{1/2}$, for a square plate with hard simple support condition ($\nu = 0.3$)

Thickness ratio, h/b	No. of terms $L \times M \times N$	Symmetry class and mode number SS-1	SS-2	SS-3	SA-1	SA-2	SA-3	AA-1	AA-2	AA-3
(a) *Antisymmetric thickness modes*										
0.01	$4\times4\times3$	1.9993	9.9847	9.9847	4.9957	12.972	17.625	7.9889	20.555	20.555
	$5\times5\times4$	1.9993	9.9826	9.9826	4.9956	12.971	16.971	7.9888	19.950	19.950
	$5\times5\times5$	1.9993	9.9826	9.9826	4.9956	12.971	16.971	7.9888	19.950	19.950
	$6\times6\times4$	1.9993	9.9826	9.9826	4.9956	12.971	16.950	7.9888	19.931	19.931
	$7\times7\times4$	1.9993	9.9826	9.9826	4.9956	12.971	16.950	7.9888	19.930	19.930
0.25	$4\times4\times3$	1.6830	5.8959	5.8959	3.5524	7.0502	8.5263	5.0359	9.4121	9.4121
	$5\times5\times4$	1.6830	5.8952	5.8952	3.5524	7.0497	8.4110	5.0359	9.3274	9.3274
	$5\times5\times5$	1.6830	5.8952	5.8952	3.5524	7.0497	8.4110	5.0359	9.3274	9.3274
	$6\times6\times4$	1.6830	5.8952	5.8952	3.5524	7.0497	8.4072	5.0359	9.3246	9.3246
	$7\times7\times4$	1.6830	5.8952	5.8952	3.5524	7.0497	8.4072	5.0359	9.3246	9.3246
(b) *Symmetric thickness modes*										
0.25	$4\times4\times3$	3.6902	6.1720	8.2522	2.6094	5.8348	7.8282	5.2187	5.2187	7.3806
	$5\times5\times4$	3.6902	6.1720	8.2515	2.6094	5.8347	7.8281	5.2187	5.2187	7.3804
	$5\times5\times5$	3.6902	6.1720	8.2515	2.6094	5.8347	7.8281	5.2187	5.2187	7.3804
	$6\times6\times4$	3.6902	6.1720	8.2515	2.6094	5.8347	7.8281	5.2187	5.2187	7.3804
	$7\times7\times4$	3.6902	6.1720	8.2515	2.6094	5.8347	7.8281	5.2187	5.2187	7.3804

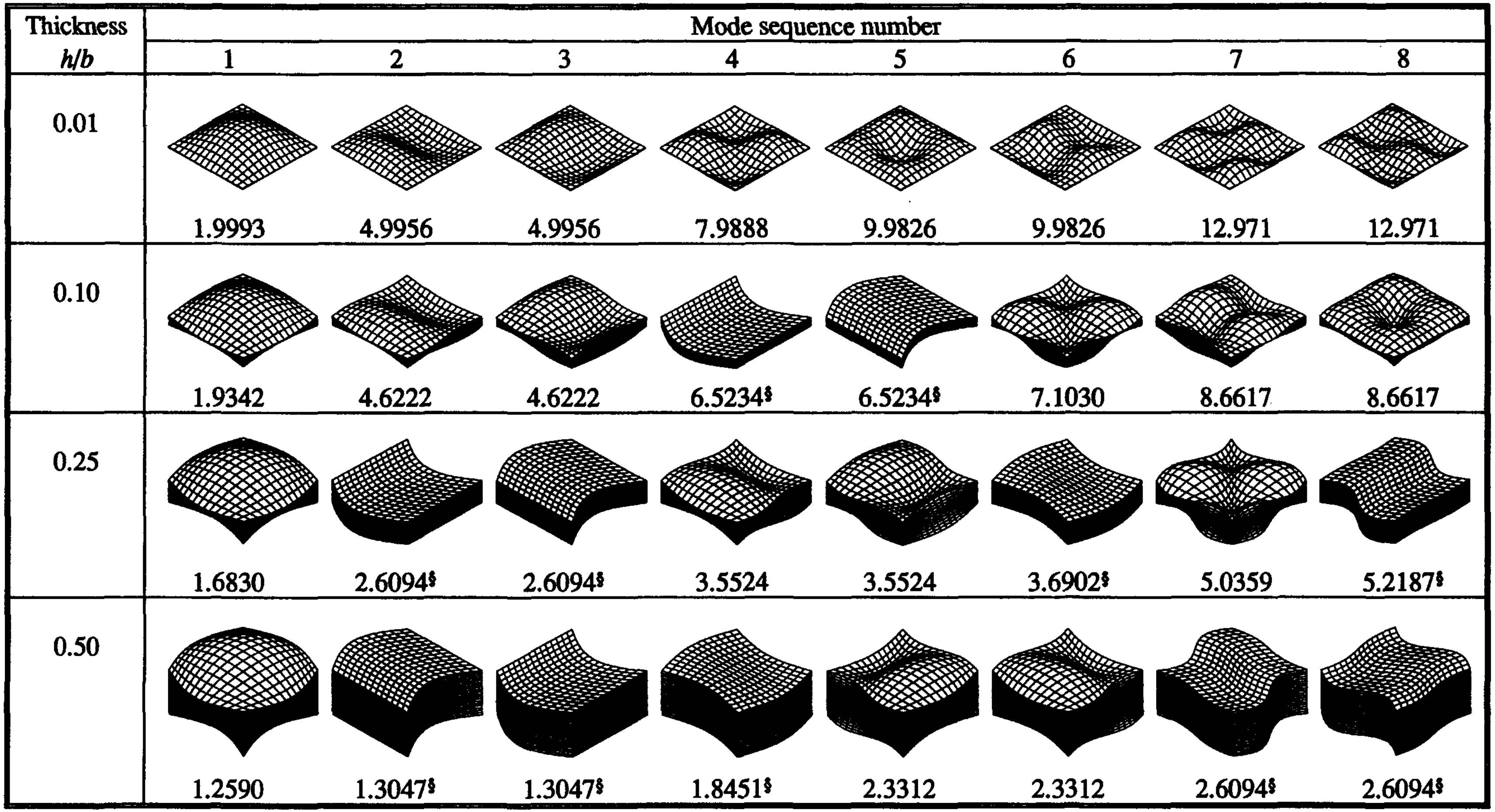
Table 2(b). Convergence of frequency parameters, $\lambda = \omega b^2/\pi^2(\rho h/D)^{1/2}$, for a square plate with soft simple support condition ($\nu = 0.3$)

Thickness ratio, h/b	No. of terms $L \times M \times N$	Symmetry class and mode number								
		SS-1	SS-2	SS-3	SA-1	SA-2	SA-3	AA-1	AA-2	AA-3
(a) *Antisymmetric thickness modes*										
0.01	$4 \times 4 \times 3$	1.9975	10.022	10.022	4.9925	13.027	17.620	7.9810	20.538	20.538
	$5 \times 5 \times 4$	1.9967	9.9789	9.9790	4.9910	12.957	16.966	7.9770	19.931	19.931
	$5 \times 5 \times 5$	1.9967	9.9789	9.9790	4.9910	12.957	16.966	7.9770	19.931	19.931
	$6 \times 6 \times 4$	1.9957	9.9759	9.9759	4.9895	12.949	16.943	7.9727	19.905	19.905
	$7 \times 7 \times 4$	1.9947	9.9741	9.9741	4.9878	12.943	16.940	7.9684	19.898	19.898
	$8 \times 8 \times 4$	1.9936	9.9722	9.9722	4.9862	12.938	16.939	7.9643	19.891	19.891
	$9 \times 9 \times 4$	1.9927	9.9706	9.9707	4.9848	12.933	16.937	7.9609	19.886	19.886
0.25	$4 \times 4 \times 3$	1.5708	5.7930	5.8283	3.4366	6.8599	8.4534	4.8327	9.2040	9.2815
	$5 \times 5 \times 4$	1.5705	5.7916	5.8269	3.4361	6.8574	8.3419	4.8315	9.1284	9.1953
	$5 \times 5 \times 5$	1.5705	5.7916	5.8269	3.4361	6.8574	8.3419	4.8315	9.1284	9.1953
	$6 \times 6 \times 4$	1.5705	5.7916	5.8268	3.4361	6.8571	8.3381	4.8314	9.1252	9.1921
	$7 \times 7 \times 4$	1.5705	5.7915	5.8267	3.4360	6.8570	8.3380	4.8313	9.1250	9.1920
	$8 \times 8 \times 4$	1.5705	5.7915	5.8267	3.4360	6.8570	8.3380	4.8313	9.1250	9.1919
(b) *Symmetric thickness modes*										
0.25	$4 \times 4 \times 3$	3.6902	4.2407	4.8202	3.4931	5.2362	7.0294	3.2605	6.0980	7.3806
	$5 \times 5 \times 4$	3.6902	4.2388	4.8197	3.4921	5.2356	7.0202	3.2604	6.0948	7.3804
	$5 \times 5 \times 5$	3.6902	4.2388	4.8197	3.4921	5.2356	7.0202	3.2604	6.0948	7.3804
	$6 \times 6 \times 4$	3.6902	4.2381	4.8196	3.4918	5.2355	7.0195	3.2604	6.0941	7.3804
	$7 \times 7 \times 4$	3.6902	4.2378	4.8196	3.4917	5.2354	7.0192	3.2603	6.0938	7.3804
	$8 \times 8 \times 4$	3.6902	4.2376	4.8195	3.4916	5.2354	7.0190	3.2603	6.0937	7.3804

Table 3. Frequency parameters, $\lambda = \omega b^2/\pi^2(\rho h/D)^{1/2}$, for a square plate with different support conditions ($\nu = 0.3$)

Thickness	Symmetry classes and mode sequence number								
h/b	SS-1	SS-2	SS-3	SA-1†	SA-2†	SA-3†	AA-1	AA-2	AA-3
(a) *Hard simple support condition* ($\sigma_n = 0$, $u_t = 0$ and $u_3 = 0$)									
0.01	1.9993	9.9826	9.9826	4.9956	12.971	16.950	7.9888	19.930	19.930
0.10	1.9342	8.6617	8.6617	4.6222	6.5234‡	10.879	7.1030	13.047 ‡	13.047 ‡
0.25	1.6830	3.6902‡	5.8952	2.6094‡	3.5524	5.8347‡	5.0359	5.2187‡	5.2187‡
0.50	1.2590	1.8451‡	2.9325‡	1.3047‡	2.3312	2.9174‡	2.6094‡	2.6094‡	3.1080
(b) *Soft simple support condition* ($\sigma_n = 0$, $\sigma_t = 0$ and $u_3 = 0$)									
0.01	1.9927	9.9706	9.9707	4.9848	12.933	16.937	7.9609	19.886	19.886
0.10	1.8588	8.5535	8.5684	4.5155	8.6992‡	10.597	6.8629	8.1476‡	15.163 ‡
0.25	1.5705	3.6902‡	4.2376‡	3.4360	3.4916‡	5.2354‡	3.2603‡	4.8313	6.0937‡
0.50	1.1752	1.8451‡	2.1373‡	1.7538‡	2.2533	2.6094	1.6312‡	2.6094	2.9965

† For a square plate with homogeneous boundary condition, the frequencies for SA and AS modes are identical.
‡ Symmetric thickness modes (surface parallel motions).



Thickness h/b	Mode sequence number 1	2	3	4	5	6	7	8
0.01	1.9993	4.9956	4.9956	7.9888	9.9826	9.9826	12.971	12.971
0.10	1.9342	4.6222	4.6222	6.5234§	6.5234§	7.1030	8.6617	8.6617
0.25	1.6830	2.6094§	2.6094§	3.5524	3.5524	3.6902§	5.0359	5.2187§
0.50	1.2590	1.3047§	1.3047§	1.8451§	2.3312	2.3312	2.6094§	2.6094§

Fig. 2. Three-dimensional mode shapes of a hard simply supported plate at different thickness ratios (§ symmetric thickness modes).

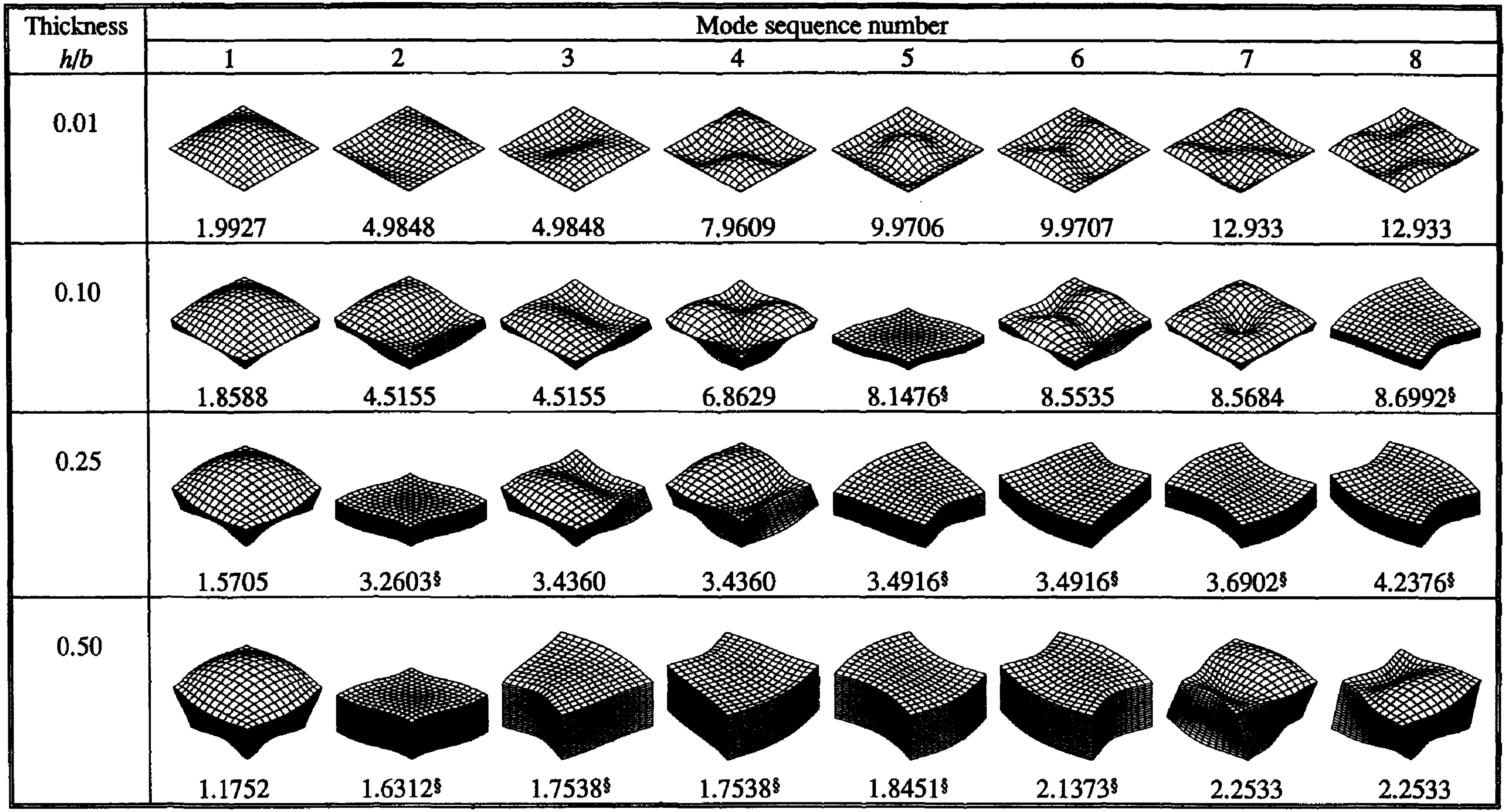


Fig. 3. Three-dimensional mode shapes of a soft simply supported plate at different thickness ratios (§ symmetric thickness modes).

The eigenvalue equation derived earlier can be rewritten in the following form :

$$\begin{bmatrix} \mathbf{K}_A & \mathbf{K}_B \\ \mathbf{K}_B^{\mathrm{T}} & \mathbf{K}_{33}-\Omega^2\mathbf{M}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{C}^{12} \\ \mathbf{C}^3 \end{Bmatrix} = \begin{Bmatrix} \{\mathbf{0}\} \\ \{\mathbf{0}\} \end{Bmatrix} \tag{21}$$

where

$$\mathbf{K}_A = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12}^{\mathrm{T}} & \mathbf{K}_{22} \end{bmatrix} \tag{22a}$$

$$\mathbf{K}_B = \begin{bmatrix} \mathbf{K}_{13} \\ \mathbf{K}_{23} \end{bmatrix} \tag{22b}$$

and

$$\{\mathbf{C}^{12}\} = \begin{Bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{Bmatrix}. \tag{22c}$$

Equation (21) can be expanded as follows

$$\mathbf{K}_A\{\mathbf{C}^{12}\}+\mathbf{K}_B\{\mathbf{C}^3\} = \{\mathbf{0}\} \tag{23}$$

$$\mathbf{K}_B^{\mathrm{T}}\{\mathbf{C}^{12}\}+(\mathbf{K}_{33}-\Omega^2\mathbf{M}_{33})\{\mathbf{C}^3\} = \{\mathbf{0}\}. \tag{24}$$

From eqn (23), we have

$$\{\mathbf{C}^{12}\} = -\mathbf{K}_A^{-1}\mathbf{K}_B\{\mathbf{C}^3\}. \tag{25}$$

Substituting eqn (25) into (24) results in a more condensed form of the eigenvalue equation

$$(\hat{\mathbf{K}}-\Omega^2\hat{\mathbf{M}})\{\mathbf{C}^3\} = \{\mathbf{0}\}. \tag{26}$$

The equivalent stiffness and mass matrices are given by

$$\hat{\mathbf{K}} = -\mathbf{K}_B^{\mathrm{T}}\mathbf{K}_A^{-1}\mathbf{K}_B+\mathbf{K}_{33} \tag{27a}$$

$$\hat{\mathbf{M}} = \mathbf{M}_{33}. \tag{27b}$$

The condensed eigenvalue equation is used to compute the frequency parameters for the antisymmetric thickness modes. The in-plane inertia in these modes are analogous to the rotary inertia effects in the Mindlin plate formulation. Figures 4 and 5 give the frequency parameters obtained with and without considering the in-plane inertia. It is observed that for small thickness ratios ($h/b \leqslant 0.01$), both approaches yield very similar frequency solutions. In other words, the influence of in-plane inertia (or the rotary inertia in the context of Mindlin plate theory) is not significant at small thickness dimensions. The deviation began to widen as the plate thickness increased. This is predictable as these inertia effects are believed to be more significant in the vibration of thick plates. Nevertheless, it is deduced from Figs 4 and 5 that the frequency solutions obtained from eqn (26) are closer to that obtained from eqn (7) for plates with thickness ratios $h/b \leqslant 0.3$. It is worth noting that the determinant size in the condensed eigenvalue equation is only one-third of that of

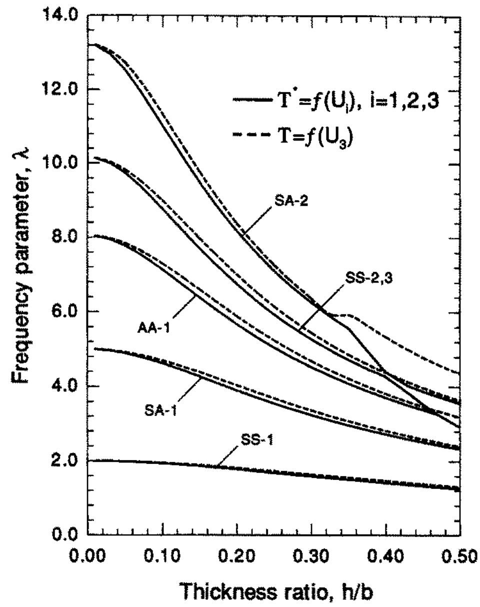


Fig. 4. Plots of frequency parameters versus thickness ratios for plates with hard simply supported boundary conditions with and without consideration of in-plane inertia (* denotes maximum kinetic energy).

the complete form as defined in eqn (7). This can be translated to more than 50.0% saving in terms of the computational effort required for the extraction of frequency solutions.

5. CONCLUSIONS

A comprehensive study of the three-dimensional normal mode behaviours of simply supported thick plates was presented. The global three-dimensional Ritz continuum approach developed on the basis of a three-dimensional elasticity theory was employed for solutions. In this study, we investigate the influence of both the hard and soft simple support conditions upon the overall vibration responses of thick plates. A numerical experiment on

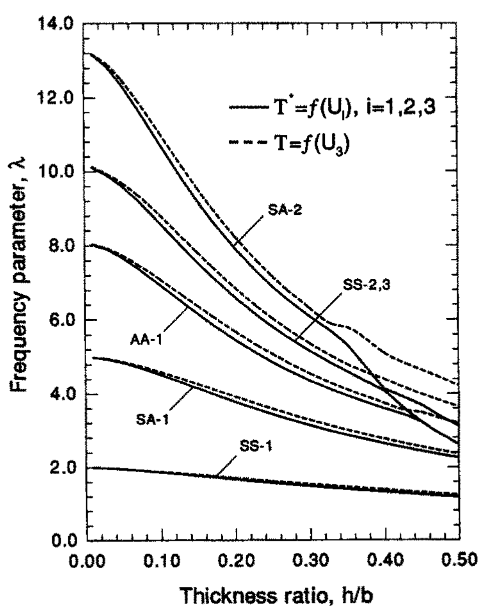


Fig. 5. Plots of frequency parameters versus thickness ratios for plates with soft simply supported boundary conditions with and without consideration of in-plane inertia (* denotes maximum kinetic energy).

the influence of in-plane inertia on the vibration frequencies was also carried out. Some observations can be drawn directly from the present studies.

- The difference between the solutions for hard and soft simple support conditions becomes negligible for thin plates. As the thickness increases, it is found that rectangular plates with hard support conditions vibrate at higher natural frequencies.
- For thin plates, the antisymmetric thickness modes (which exhibit out-of-plane motions) dominate the lower vibration spectrum for both hard and soft simply supported plates. As the thickness increases, the symmetric thickness modes (which include thickness–twist, thickness–shear and surface parallel motions) begin to precede some of the out-of-plane modes.
- The determinant size of the eigenvalue problem can be significantly reduced by ignoring the contributions of the in-plane components to the kinetic energy. The error resulting from such a simplification is found to be small for plate with thickness ratio $h/b \leqslant 0.30$. This error becomes significant only at higher modes and for thicker plates.

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